On the Accuracy of Perfectly Matched Layers using a Finite Element Formulation *

Michael R. Lyons, Anastasis C. Polycarpou and Constantine A. Balanis

Department of Electrical Engineering
Telecommunications Research Center
Arizona State University
Tempe, AZ 85287-7206

ABSTRACT

A Perfectly Matched Layer (PML) is applied to a three-dimensional edge-based finite element formulation to calculate the S-parameters of waveguide structures. The PML region is implemented in the finite element code as a non-physical uniaxial anisotropic lossy material. Numerical results demonstrate the accuracy and future potential of such an absorber.

I Introduction

Since the introduction of the Perfectly Matched Layer (PML) concept by Berenger [1], much effort was mainly concentrated on applying this boundary condition to a variety of problems such as scattering, antenna radiation, and MMIC structures. The implementation of the PML absorber in finite difference methods showed incredible improvement in reducing the reflection error caused by the truncated mesh. Published results illustrate cases where the reflection error due to the PML region is of the order of -100 dB or even smaller, which creates new possibilities for highly accurate simulations using reasonable computer resources.

The original implementation of the PML by Berenger [1] was formulated using a split-field approach which is not governed by Maxwell's equations. In addition, such an approach is suitable only for the finite-difference time-domain method. Recently, it was shown by Gedney [2] that the PML region can be equivalently modeled as a uniaxial anisotropic lossy material. However, like most other contributions on the PML absorbers, Gedney's work was also concentrated on the finite-difference time-domain technique.

This paper basically formulates the PML absorber following similar guidelines as those presented by Gedney [2] but applied to the finite element method. The main idea is to treat the PML absorber as a non-physical uniaxial electric and magnetic anisotropic material. In addition, the PML material is highly lossy so that the incident field is significantly attenuated before it actually reaches the terminating perfect conducting wall. It was observed by Gedney that this approach is straightforward and provides exactly the same accuracy as the original Berenger boundary conditions. It is also worth mentioning here that the concept of the Perfectly Matched Layer, although a different approach from this paper, has been applied recently by Pekel and Mittra [3] in calculating radar cross sections of conducting plates. However, the obtained results were not as accurate compared to data obtained using the Method of Moments.

II Analysis

The finite element formulation starts with the discretization of the electric field vector equation:

$$\nabla \times \left([\mu_r]^{-1} \cdot \nabla \times \mathbf{E} \right) - k_\circ^2 \; [\epsilon_r] \; \mathbf{E} = 0 \qquad (1)$$

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where $[\mu_r]$ and $[\epsilon_r]$ are tensors. As far as the waveguide problem is concerned, the appropriate boundary conditions on the surface walls are either Dirichlet

$$\hat{n} \times \mathbf{E} = 0 \tag{2}$$

for perfect conductors or mixed

$$\hat{n} \times (\nabla \times \mathbf{E}) + jk_{z10}\hat{n} \times (\hat{n} \times \mathbf{E}) = -2jk_{z10}\mathbf{E}^{inc} \tag{3}$$

at the input port. Note that k_{z10} is the propagation constant of the dominant mode and $\mathbf{E}^{inc}$ is the incident field. It was assumed that the incident wave is propagating in the z-direction.

According to Gedney's paper, the Perfectly Matched Layer can be modeled as a uniaxial anisotropic material characterized by the following permittivity and permeability tensors:

$$[\epsilon]^{pml} = [\mu]^{pml} = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1/K \end{bmatrix} \tag{4}$$

where K is given by

$$K = 1 - j\frac{\sigma}{\omega\epsilon_o\epsilon_r} \tag{5}$$

It is important to emphasize that the permittivity and permeability tensor given in (4) would provide a perfectly matched layer only for an incident wave traveling in the z-direction. Also, from (5) it is observed that the PML region is lossy; the actual loss is controlled by the value of σ. In the results presented by Berenger, it was clearly pointed out that the mismatch at the PML interface can be significantly reduced by carefully selecting the conductivity, σ, of the material. It is usually the case that higher reflections occur when the transition from one medium to another becomes more abrupt. A judicious choice for σ is given in [2]

$$\sigma(z) = \sigma_{max}\left(\frac{z - z_o}{d}\right)^m \tag{6}$$

where σ_{max} is the maximum value of the material conductivity, d is the depth of the PML region, m is the order of the spatial polynomial, and z_o is the position of the interface.

III Results

The numerical accuracy of the anisotropic PML was investigated by computing the S-parameters of waveguide structures similar to the one illustrated in Fig. 1. The rectangular waveguide is excited with the TE_{10} mode at the input port whereas the output port is terminated with a 5-layer PML medium backed with a perfect conductor. The waveguide region is discretized with tetrahedral elements using the mesh generator of SDRC-IDEAS. Initially, we considered the waveguide structure shown in Fig. 1 with $b = 1$ cm, but without the dielectric discontinuity. The S_{11} is calculated within a broadband of 10 GHz for two different discretizations: Case #1 with approximately 6000 unknowns and Case #2 with approximately 13000 unknowns. In Case #1, each layer in the PML region, on the average, can accomodate only one tetrahedral element in the longitudinal direction, whereas in Case #2, each layer can accomodate two tetrahedral elements. Also, in both cases σ_{max} was chosen to be equal to 4 (note that the dimensions of the waveguide are in centimeters) and the spatial polynomial order m was also set to 4. The corresponding results are illustrated in Fig. 2. It is observed than in Case #1 the total reflection error is always less than -30 dB whereas in Case #2 the total reflection error is always less than -40 dB. Referring to this figure it is important to emphasize the following:

- The calculated S_{11} does not include only the reflections from the PML region but also the finite element discretization error which is widely known to be of order h^2, where h is the maximum edge length of the tetrahedral element used in the mesh.
- The finer the discretization the lower the total reflection error. The reason is attributed both to a smaller discretization error and a better field representation in the PML region.
- An ineffective PML region would most likely result in an error close to 0 dB because of the presence of the perfect conducting wall at the output port. In our case it looks like

the PML region acts as a very good absorber within a broad frequency band.

The second case considered was the same rectangular waveguide shown in Fig. 1 but with the dielectric discontinuity in place. The dielectric constant for the discontinuity is $\epsilon_r = 6$. As was the case before, the waveguide is terminated with a 5-layer PML medium backed with a perfect conductor. The number of unknowns in the finite element region is approximately equal to 13000. The S_{11} is computed for values of $k_o b$ between 1.6 to 3.0. Our results are compared with data obtain from Ise *et. al.* [4]. The comparison between the two data sets is depicted in Fig. 3. It is clearly illustrated that the magnitude of S_{11} obtained using the Perfectly Matched Layer is in excellent agreement with the data extracted from the paper published by Ise *et. al.*.

The last case we considered in verifying the effectiveness of the Perfectly Matched Layer as applied to the finite element method was to calculate the $|S_{21}|^2$ of a perfectly conducting right-angle bend shown in Fig. 4. The dimensions of the waveguide were chosen so that $a = 2b$. The output port of the right-angle bend was again terminated with a 5-layer PML medium backed with a perfect conductor. Note that the orientation of the PML medium is changed; its permittivity and permeability tensors should be modified appropriately so it can absorb incident waves that are traveling in the positive x-direction instead of the z-direction. The size of the problem is approximately equal to 14000 unknowns. Again, our results are compared with data extracted from the paper published by Ise *et. al.* for values of $\frac{2a}{\lambda}$ between 1.0 and 2.0. An excellent agreement between the two sets of data is illustrated in Fig. 5.

IV Conclusions

A Perfectly Matched Layer made out of a lossy uniaxial anisotropic material was effectively implemented into a three-dimensional vector finite element formulation for the calculation of the S-parameters of various waveguide structures. Numerical results demonstrate that such an absorber

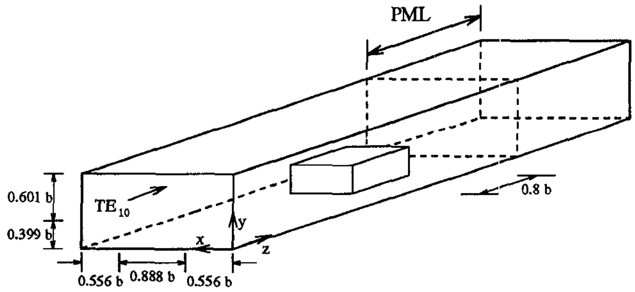

Figure 1: Dielectric-loaded waveguide terminated with a perfectly matched layer. The relative permittivity of the dielectric discontinuity is $\epsilon_r = 6$.

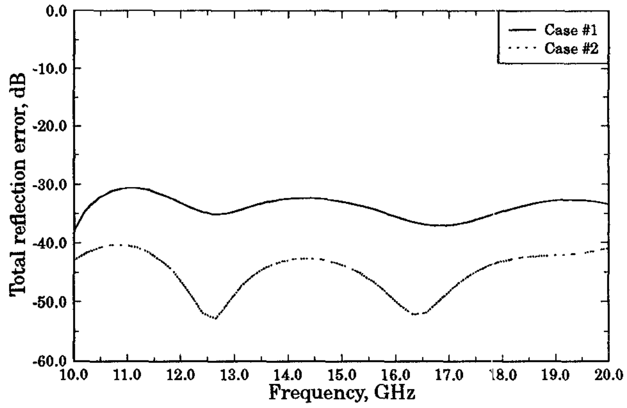

Figure 2: Air-filled waveguide ($b = 1$ cm) terminated with a perfectly matched layer. The PML region is backed with a perfectly conducting wall.

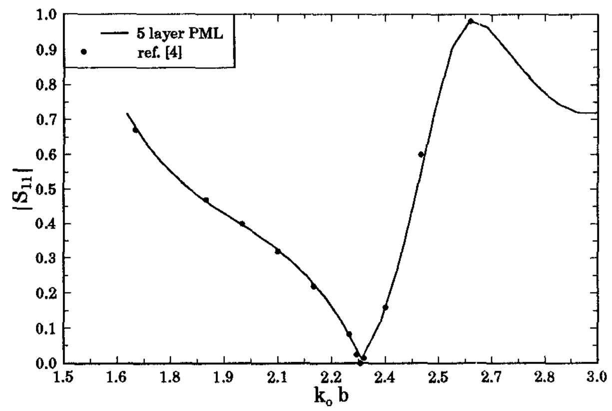

Figure 3: Dielectric loaded waveguide ($b = 1$ cm) terminated with a perfectly matched layer. The PML region is backed with a perfectly conducting wall and the relative permittivity of the dielectric discontinuity is $\epsilon_r = 6$.

can be succesfully and accurately used to terminate finite element meshes. However, more investigations on how to improve the PML need to be continued.

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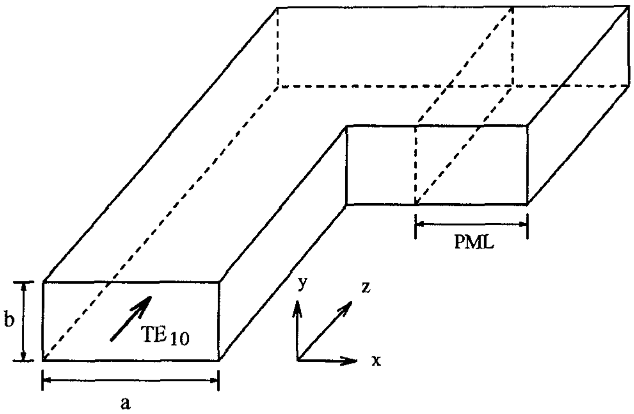

Figure 4: Air-filled right-angle bend terminated with a PML medium at the output port.

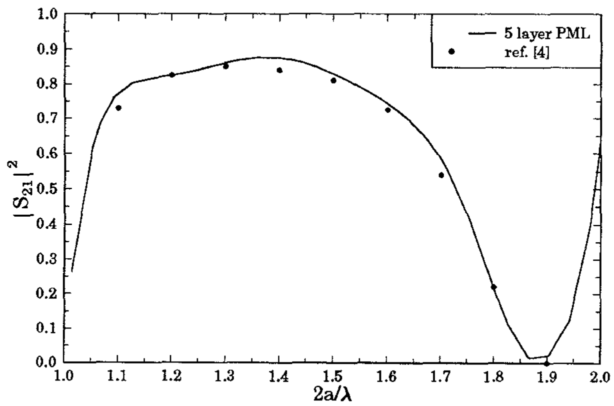

Figure 5: Air-filled right-angle bend ($a = 2b$) terminated with a perfectly matched layer at the output port. The PML region is backed with a perfectly conducting wall.